Note, we present and discuss example calculations of 3DES feedback extremals for a contemporary high-performance aircraft. In addition to being required for the guidance law, the 3DES extremals are of interest in their own right, both as numerical approximations to optimal trajectories and as a means of gaining insight into the nature of high-performance-aircraft optimal maneuvers.

Three-Dimensional Energy-State Extremals

The 3DES dynamic model is

$$\dot{E} = P(E, V, n, \beta), \quad \dot{\chi} = gn \sin\theta / V$$

$$\dot{x} = V \cos\chi, \quad \dot{y} = V \sin\chi, \quad n \cos\theta = 1$$

where $(\dot{\ }) = \mathrm{d}(\ )/\mathrm{d}t$, and the state variables are horizontal position (x,y), heading angle (χ), and specific energy (E); P is specific excess power, and n is load factor. These equations are valid in any inertial reference frame.

This dynamic model results from assuming altitude and flight-path angle to be fast variables, compared with E, χ, x, and y. Many other time-scale separations of the flight dynamics equations have been investigated; these are critically reviewed in Ref. 3. The first systematic study of three-dimensional flight-path optimization problems by singular perturbations was performed by Kelley,[4] and other noteworthy approaches may be found in Refs. 5 and 6.

In Ref. 7 we developed necessary conditions for the optimal controls β (throttle), θ (bank angle), and V (speed) for the minimum time transfer on the 3DES dynamic model and gave example extremals for a specific aircraft. These extremals are open-loop; they were produced by backward integration of the necessary conditions and are thus parameterized in terms of final conditions E_f and χ_f.

The extremals to be discussed in this Note differ in two important aspects from those of Ref. 7. First, they are closed-loop in the sense that iterative backward integrations have been performed to match specified initial conditions. Consequently, these extremals are parameterized in terms of initial conditions E_0 and χ_0, as well as trajectory duration T. Second, the extremals are computed for a current-generation aircraft, a version of the F-15, whereas those of Ref. 7 are for a version of the F-4.

Time histories of example extremal trajectories in feedback form for the F-15 aircraft are shown in Fig. 1. Two of the control variables are shown (V and ω, the latter being a ratio of bank angle to maximum available bank angle, as limited by maximum load factor and stall) and two state variables (E and χ). All trajectories have values of $E_0 = 50 \times 10^3$ ft and $T = 30$ s.

In forward time ($t = T - \tau$), Fig. 1a shows that trajectories with χ_0 greater than approximately 75 deg start on the corner velocity locus (locus of maximum instantaneous turn rate). They then make successive transitions to two higher unbounded speeds and finally end on the maximum speed bound. Trajectories with $\chi_0 < 75$ deg begin with speeds greater than the corner velocity and end at maximum speed. Figure 1b shows that trajectories with $\chi_0 > 25$ deg begin with full bank and then smoothly transition to zero bank at termination. For $\chi_0 < 25$ deg the trajectories start at less than full bank. It was also found that, for χ_0 greater than approximately 150 deg, the throttle setting started at zero and subsequently switched to full but that for $\chi_0 < 150$ deg the throttle is set at full throughout.

The paths of the extremal trajectories in the horizontal plane are shown in Fig. 2. The particular inertial reference frame chosen to display these results has its origin at trajectory termination; this coordinate system has certain advantages in pursuit-evasion.[7] (Note that the x- and y-axis scales are different.) The characteristic of hard-turning transitioning into a high-speed dash is apparent.

Figure 3 shows the trajectories in the flight envelope. Here we see clearly the behavior of trajectories starting at large heading angles: after an initial period of hard turning on the corner velocity locus, the paths transition to the vicinity of the energy climb path, where they remain briefly. They then jump to a speed near the maximum speed and finally transition to the maximum speed for the final dash. The energy climb path is the locus of most rapid energy increase in vertical flight, and the extremal control is apparently using this property to briefly accumulate energy. In fact, the period of time spent near the energy climb path coincides with the boost in energy between 20 and 25 s exhibited on Fig. 1c. The time spent on these branches of the reduced solution will, of course, be greatly reduced when boundary-layer corrections (fast dynamics) are included.

Concluding Remarks

The trajectories generated in this analysis display features that agree with aircraft pursuit-evasion and target-interception experience. They consist primarily of segments of hard turning on the corner velocity locus and high-speed dash on the maximum speed boundary, with occasional short intervals of optimal energy accumulation. The fact that these trajectories exhibit frequent large jumps between these various branches of the 3DES solution means that the transitions between these branches, the boundary-layer motions, will be important. Even in this situation, however, the reduced solution is of key importance because in singularly perturbed systems it is the reduced solution that provides the equilibrium points that are the basis of the boundary-layer computations.

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Eigenstructure of the State Matrix of Balanced Realizations

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Introduction

THE calculation of balanced representations and their properties have been studied extensively.[1-6] One aspect of these representations, however, has not been studied; namely,

Received March 2, 1988; revision received May 16, 1988.

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what happens to the eigenvectors of the balanced system. Under a state coordinate transformation, the eigenvalues of the state matrix A are invariants, the eigenvectors are not. The input and output matrices also are not invariant under such a transformation. The balanced model has some interesting symmetry properties; symmetry of the input matrix as compared to the output matrix and symmetry of the A matrix itself. For a single-input/single-output (SISO) system it has been shown that the A matrix is parity symmetric and the input and output matrices are also related by the same parity matrix relating A to its transpose.[2,3,6] For a multiple-input/multiple-output (MIMO) system, balancing imposes some symmetry on the A matrix itself as well as on the B and C matrices. For a special class of MIMO systems, namely those with orthogonally symmetric transfer function matrices, the A matrix of the balanced representation also can be parity symmetric.[7] The aforementioned symmetries will be reflected on the right and left eigenvectors of the balanced representation. In this Note, we show that this symmetry, taken together with the symmetry imposed by balancing on the B and C matrices, is bound to show in the alignment of the left eigenvectors and the columns of the B matrix and the right eigenvectors and the rows of the C matrix.

SISO-Balanced Representation

Let us consider a state-space representation

$$\dot{x} = Ax + bu \tag{1a}$$

$$y = cx \tag{1b}$$

where A is $n \times n$, b and $c^T \epsilon R^n$. The A matrix is assumed to have distinct eigenvalues, and its eigenstructure is not ill-conditioned. Let

$$Ap_i = \lambda_i p_i \tag{2}$$

$$A^T q_i = \lambda_i q_i \tag{3}$$

where λ_i is the ith eigenvalue of A, and p_i and q_i are the corresponding right and left eigenvectors. Let us define the angle between the ith left eigenvector and b as

$$\cos\theta(q_i, b) = \frac{|q_i^T b|}{\|q_i\| \|b\|} \tag{4}$$

and the angle between the ith right eigenvector and c as

$$\cos\theta(p_i, c^T) = \frac{\|cp_i\|}{\|p_i\| \|c^T\|} \tag{5}$$

Lemma 1: In a balanced SISO representation, the angle between the ith left eigenvector and the column vector b is the same as the angle between the ith right eigenvector and the column vector c^T.

Proof: Moore[1] noted that each of the SISO-balanced systems that he tested had an A matrix with absolute value symmetry. Later on it was proved[2,3,6] that a balanced SISO system has an A matrix, which is either parity symmetric (i.e., $A^T = EAE$, where E is a diagonal matrix with entries ± 1) or it can be made parity symmetric through an orthogonal transformation and b and c vectors such that $b = Ec^T$. Replacing A^T in Eq. (3) with EAE yields

$$A^T q_i = EAEq_i = \lambda_i q_i$$

which, on multiplication from the left with E, gives

$$A(Eq_i) = \lambda_i (Eq_i) \tag{6a}$$

or

$$p_i = Eq_i \tag{6b}$$

Since $b = Ec^T$, then $|q_i^T b| = |cp_i|$ and $\|b\| = \|c^T\|$. Hence,

$$\cos\theta(q_i, b) = \cos\theta(p_i, c^T) \tag{7}$$

If the original balanced representation is not parity symmetric, then it can be made so using an orthogonal transformation.[5] Since orthogonal transformations preserve angles and lengths, Eq. (7) continues to hold in this case.

Example 1

Example 1 in Moore's paper[1] is as follows:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -50 & -79 & -33 & -5 \end{bmatrix}$$

$$b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \qquad c = [50\ 15\ 1\ 0]$$

Whereas the angles $\theta(q_i, b)$ and $\theta(p_i, c^T)$ of the unbalanced system are not equal, those of the balanced realizations are equal, as shown in Table 1.

Example 2

Example 2 in Moore's paper[1] is as follows:

$$A = \begin{bmatrix} 0 & 0 & 0 & -150 \\ 1 & 0 & 0 & -245 \\ 0 & 1 & 0 & -113 \\ 0 & 0 & 1 & -19 \end{bmatrix}$$

$$b = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \qquad c = [0\ 0\ 0\ 1]$$

Again, whereas the angles $\theta(q_i, b)$ and $\theta(p_i, c^T)$ of the unbalanced system are not equal, those of the balanced realizations are equal, as shown in Table 2.

Table 1 Comparison of the angles obtained using the balanced and unbalanced representations for Moore's Example 1

	Unbalanced representation		Balanced representation	
Mode	$\cos\theta\ (q_i, b)$	$\cos\theta\ (p_i, c^T)$	$\cos\theta\ (q_i, b)$	$\cos\theta\ (p_i, c^T)$
$-1+j5$	0.0512	0.0097	0.4232	0.4232
$-1-j5$	0.0512	0.0097	0.4232	0.4232
-1	0.0172	0.3448	0.8100	0.8100
-2	0.0271	0.0499	0.6283	0.6283

Table 2 Comparison of the angles obtained using the balanced and unbalanced representations for Moore's Example 2

	Unbalanced representation		Balanced representation	
Mode	$\cos\theta\ (q_i, b)$	$\cos\theta\ (p_i, c^T)$	$\cos\theta\ (q_i, b)$	$\cos\theta\ (p_i, c^T)$
-1	0.3638	0.0056	0.8903	0.8903
-3	0.0085	0.0120	0.8500	0.8500
-5	0.0019	0.0184	0.5885	0.5885
-10	0.0014	0.0346	0.525	0.5252

MIMO-Balanced Representation

Let us consider a MIMO representation

$$\dot{x} = Ax + Bu \quad (8a)$$

$$y = Cx \quad (8b)$$

where B is $n \times m$ and C is $\ell \times n$. We restrict our system to have an orthogonally symmetric transfer function. Furthermore, let us assume that the representation is balanced so that A is a parity symmetric matrix.[7] For this class of MIMO systems, we can state the following results.

Lemma 2: For a MIMO-balanced system representation with a parity symmetric A matrix, the B and C matrices satisfy the following relation:

$$\|B^T q_i\| = \|Cp_i\| \quad (9)$$

Proof: The controllability and observability Grammians of representation (8) are equal and diagonal.[1] The Grammian Σ^2 satisfies the following Lyapunov equations:

$$A\Sigma^2 + \Sigma^2 A^T = -BB^T \quad (10)$$

$$A^T\Sigma^2 + \Sigma^2 A = -CC^T \quad (11)$$

Premultiplying Eq. (10) by q_i^T and postmultiplying it with q_i, and premultiplying Eq. (11) by p_i^T and postmultiplying by p_i and using Eq. (6), we obtain the desired result.

Concluding Remarks

The significance of Lemma 1 is that the angles between the left eigenvectors and b can be used as measures of modal controllability, and the angles between the right eigenvectors and c can serve as measures of modal observability.[8] As a result of Lemma 1, one can say that balancing makes not only the state directions but also the modes equally controllable and observable. Actually, every direction in the state space is equally controllable and observable.

Lemma 2 indicates that B^T magnifies q_i to the same degree that C magnifies p_i for the MIMO system considered; $\|B^T q_i\|$ depends on the alignment between the left eigenvector q_i and the singular directions of B. If q_i lies in the subspace of the singular directions of B corresponding to the zero singular values, then $\|B^T q_i\|$is zero and the mode is not controllable. Thus, $\|B^T q_i\|$ serves as a measure of gross controllability of the ith mode from all inputs. A parallel argument applies for $\|Cp_i\|$ and modal observability.[8] In this case, balancing makes the modes equally controllable and observable.

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Book Announcements

WIESEL, W. E., *Space Flight Dynamics,* McGraw-Hill, New York, 1989, 323 pages.

Purpose: This book presents the newest developments and technologies in the field—all presented with an eye toward the 1990's.

Contents: Particle dynamics; two-body problem; Earth satellite operations; rigid-body dynamics; satellite attitude dynamics; gyroscopic instruments; rocket performance; re-entry dynamics; space environment; restricted problem; interplanetary trajectories; appendices.

INMAN, D. J., ***Vibration with Control, Measurement, and Stability,*** Prentice-Hall, Englewood Cliffs, NJ, 1989, 384 pages.

Purpose: This book contains a variety of new topics reflecting some changes in vibration analysis, measurement, and design that have occured in the past 20 years. Both experimental and modal analysis are interrelated and covered in depth.

Contents: Single degree of freedom systems; lumped parameter models; matrices and the free response; stability; forced response of lumped parameter systems; design considerations; control of vibrations; modal testing; distributed parameter models; formal methods of solution; operators and the free response; forced response and control; approximation of distributed parameter models; appendices.

SHABANA, A. A., ***Dynamics of Multibody Systems,*** Wiley, New York, 1989, 470 pages.

Purpose: This book presents the kinematics and dynamics of rigid and deformable bodies. The treatment covers a wide range of applications, including robotics, complex machine design, structural design, and motion analysis.

Contents: Introduction; reference kinematics; analytical techniques; mechanics of deformable bodies; classical approximation methods; finite-element formulation; computer implementation.

SHEVELL, R. S., *Fundamentals of Flight,* 2nd ed., Prentice-Hall, Englewood Cliffs, NJ, 1989, 416 pages.

Purpose: This book provides the reader with an introduction to the science and engineering of heavier-than-air flight vehicles. This edition includes a chapter on hypersonic flow.

Contents: A brief history of aeronautics; anatomy of the airplane; aerodynamic forces: dimensional analysis; theory and experiment wind tunnels; atmosphere; one-dimensional incompressible and compressible flows; two-dimensional flows; finite wing; effects of viscosity; drag calculations; aerodynamic performance; stability and control; propulsion; structures; hypersonic flow; rocket trajectories and orbits; appendices.

Errata

Markov Reliability Models for Digital Flight Control Systems

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and
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[JGCD 12, 209–219 (1989)]

EQUATION (3) on page 214 was printed incorrectly in the published paper. The equation should appear as follows:

$$e^{Qt} = I + Qt + Q^2 \frac{t^2}{2!} + Q^3 \frac{t^3}{3!} \cdots \tag{3}$$

Optimal Terminal Maneuver for a Cooperative Impulsive Rendezvous

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[JGCD 12, 433–435 (1989)]

DURING production of the paper, two equations were inadvertently altered. On page 434, the equation following Eq. (10) should read:

$$L''(x) = -\frac{m_1}{c_1^2} e^{-x/c_1} - \frac{m_2}{c_2^2} e^{-(K-x)/c_2} < 0$$

On page 435, Eq. (11) should read:

$$\text{If } L'(p_1) \le 0 \qquad \text{then } x^* = p_2 \tag{11}$$